# Note on quasi-numerically positive log canonical divisors

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**Abstract**

We propose a subconjecture that implies the semiampleness conjecture for quasi-numerically positive log canonical divisors and prove the semiampleness in some elementary cases.

In this note, every algebraic variety is defined over the field $\mathbf{C}$ of complex numbers. We follow the terminology and notation in [10].

## 1 Introduction

**Definition 1.1.** Let $D$ be a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor on a projective variety $X$. The divisor $D$ is *numerically positive* (*nup*, for short), if $(D, C) > 0$ for every curve $C$ on $X$. The divisor $D$ is *quasi-numerically positive* (*quasi-nup*, for short), if it is nef and if there exists a union $F$ of at most countably many prime divisors on $X$ such that $(D, C) > 0$ for every curve $C \not\subseteq F$ (i.e. if $D$ is nef and if $(D, C) > 0$ for every very general curve $C$).

*Remark* 1.2. The quasi-nup divisors are the divisors "of maximal nef dimension" in the terminology of the "Eight Authors" [2].

Ambro [1] and Birkar-Cascini-Hacon-McKernan [3] reduced the famous log abundance conjecture to the termination conjecture for log flips and the semiampleness conjecture (Conjecture 2.1) for quasi-nup log canonical divisors $K_X + \Delta$, in the category of Kawamata log terminal (*klt*, for short) pairs. In Section 2 we propose a subconjecture (Subconjecture 2.3) that implies the semiampleness Conjecture 2.1.

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*Remark* 1.3. We state the history in detail. In the category of klt pairs $(X, \Delta)$, Fukuda [5] reduced the log abundance to the existence and termination of log flips, the existence of log canonical bundle formula and the semiampleness of quasi-nup log canonical divisors, by using the numerically trivial fibrations ([15], see also [2]) due to Tsuji and the semiampleness criterion ([7] and [13], see also Fujino [4]) for log canonical divisors due to Kawamata-Nakayama. Ambro [1] gave and proved the celebrated log canonical bundle formula. The existence of log flips is now the theorem [3] due to Birkar, Cascini, Hacon and McKernan. This history is along the line of Reid's philosophy stated in the famous Pagoda paper [14].

There is another approach to the semiampleness Conjecture 2.1. Let $(X, \Delta)$ be a klt pair whose log canonical divisor $K_X + \Delta$ is quasi-nup. Hacon and McKernan (Lazic [12], Theorem A.6) considered to embed $(X, \Delta)$ into some log canonical pair $(\overline{X}, \overline{\Delta})$ so that $\dim \overline{X} = \dim X + 1$ and $\overline{\Delta} \geq X$, that the log canonical divisor $K_{\overline{X}} + \overline{\Delta}$ is nef and big, that $(K_{\overline{X}} + \overline{\Delta})|_X = K_X + \Delta$ and that $\overline{X}$ is endowed with the birational contraction morphism $\phi : \overline{X} \to \overline{Y}$ that contracts the prime divisor $X (= \mathrm{Exc}(\phi))$ to some point. In Section 3, motivated by this consideration, we prove the semiampleness (Theorem 3.8) for log canonical pairs in some elementary cases.

## 2 Subconjecture for klt pairs

**Conjecture 2.1.** *Let $(X, \Delta)$ be a Kawamata log terminal pair such that $X$ is projective. If the log canonical divisor $K_X + \Delta$ is quasi-nup, then it is semiample.*

We give an approach towards the above-mentioned semiampleness conjecture in this section. The approach repeats the process of finding some $(K_X + \Delta)$-trivial curve that generates a $(K_X + \Gamma)$-extremal ray for some other klt pair $(X, \Gamma)$ and contracting this extremal ray. The process would terminate at the ample log canonical divisor. To run the process, it is important not to require the $\mathbf{Q}$-factoriality of $X$.

**Definition 2.2.** We define $\overline{NE}_{D=0}(X) := \{l \in \overline{NE}(X) | \text{the intersection number } (D, l) = 0\}$ and $\overline{NE}_{D\geq 0}(X) := \{l \in \overline{NE}(X) | (D, l) \geq 0\}$ for a $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor $D$ on $X$.

**Subconjecture 2.3.** *Let $(X, \Delta)$ be a Kawamata log terminal pair such that $X$ is projective. Suppose that the log canonical divisor $K_X + \Delta$ is not ample but quasi-nup. Then there exists an effective $\mathbf{Q}$-Cartier divisor $E$ such that the intersection number $(E, l) < 0$ for some class $l \in \overline{NE}_{K_X+\Delta=0}(X)$.*

**Procedure 2.4.** Let $(X, \Delta)$ be a Kawamata log terminal pair such that $X$ is projective. Suppose that the log canonical divisor $K_X + \Delta$ is not ample but quasi-nup. Assume the existence of an effective $\mathbf{Q}$-Cartier divisor $E$ on $X$ and a member $l$ of $\overline{NE}_{K_X+\Delta=0}(X)$ such that the intersection number $(E, l) < 0$. Let $\epsilon$ be a sufficiently small positive rational number. We can write this class $l$ in the form that $l = l_0 + l_1 + l_2 + \cdots + l_p$ ($p \geq 1$), where $l_0 \in \overline{NE}_{K_X+\Delta+\epsilon E \geq 0}(X)$ and $\mathbf{R}_+ l_i$ ($i \geq 1$) are distinct $(K_X+\Delta+\epsilon E)$-extremal rays. Then $(K_X + \Delta, l_1) = 0$, because $K_X + \Delta$ is nef and $(K_X + \Delta, l) = 0$. We consider the birational contraction morphism $\phi : X \to X_1$ of the $(K_X + \Delta + \epsilon E)$-extremal ray $\mathbf{R}_+ l_1$. Put $\Delta_1 := \phi_*(\Delta)$. We note that the Picard number $\rho(X_1) = \rho(X) - 1$, that $K_X + \Delta = \phi^*(K_{X_1} + \Delta_1)$, that $(X_1, \Delta_1)$ is Kawamata log terminal and that $K_{X_1} + \Delta_1$ is quasi-nup. Remark that we can permit each of the divisorial-contraction case and the small-contraction case, because we do not require the $\mathbf{Q}$-factoriality of $X_1$.

Procedure 2.4 relates Subconjecture 2.3 to Conjecture 2.1. The following is the main result of this section:

**Theorem 2.5.** *Subconjecture 2.3 implies Conjecture 2.1.*

*Proof.* Let $(X, \Delta)$ be a Kawamata log terminal pair such that $X$ is projective and the log canonical divisor $K_X + \Delta$ is quasi-nup. If Subconjecture 2.3 is true, then, by repeating Procedure 2.4, we obtain a Kawamata log terminal pair $(X', \Delta')$ with the birational morphism $\psi : X \to X'$ such that $K_X + \Delta = \psi^*(K_{X'} + \Delta')$ and that $K_{X'} + \Delta'$ is ample, because the Picard numbers decrease 1 by 1. □

**Corollary 2.6.** *Subconjecture 2.3 and the termination conjecture for log flips imply the log abundance conjecture for klt pairs.*

*Proof.* See Remark 1.3 and the theorem above. □

*Remark* 2.7. From the corollary above and the existense theorem [8] for extremal rational curves by Kawamata, we can say that the log abundance conjecture is the existence problem for some kind of rational curves, modulo the termination of log flips.

We show that Subconjecture 2.3 is a part of Conjecture 2.1.

**Lemma 2.8.** *Let $(X, \Delta)$ be a Kawamata log terminal pair such that $X$ is projective. Suppose that $K_X + \Delta$ is not ample but quasi-nup and semiample. Then there exists an effective $\mathbf{Q}$-Cartier divisor $E$ such that the intersection number $(E, l) < 0$ for some class $l \in \overline{NE}_{K_X+\Delta=0}(X)$.*

*Proof.* Consider the surjective morphism $\phi : X \to Y (= \Phi_{|k(K_X+\Delta)|}(X))$ induced by the linear system $|k(K_X + \Delta)|$ for a sufficiently large and divisible integer $k$. This morphism $\phi$ becomes birational, because of the Stein factorisation theorem and the fact that the pull-backs of ample divisors by finite morphisms are ample. Then $k(K_X + \Delta) = \phi^* H$ for an ample divisor $H$ on $Y$. By the Kodaira Lemma, if $m$ is sufficiently large and divisible, then $m\phi^* H = A + E$ for some ample divisor $A$ and some effective divisor $E$. For every $\phi$-exceptional curve $C$, we obtain the inequality that $(E, C) < 0$, because $(m\phi^* H, C) = 0$ and $(A, C) > 0$. Here the class $[C]$ belongs to $\overline{NE}_{K_X+\Delta=0}(X)$. □

**Proposition 2.9.** *Conjecture 2.1 implies Subconjecture 2.3.*

*Proof.* Lemma 2.8 gives the assertion. □

# 3 Log canonical pairs in some elementary cases

We prove the semiampleness for log canonical pairs in some elementary cases.

**Assumption 3.1.** Let $f : X \to Y$ be a birational morphism between normal projective varieties of dimension $n$ such that $E := \mathrm{Exc}(f)$ is a prime divisor and that $X$ and $E$ are smooth. Consider some base-point-free complete linear system on $Y$ such that its general member $H$ is irreducible or $H$ is equal to 0. Assume that $K_X + f^*H + E$ is nup.

**Proposition 3.2.** *Under Assumption 3.1. The divisor $K_X + f^*H + (1-\epsilon)E$ is nef for every small number $\epsilon > 0$.*

*Proof.* Every $K_X + f^*H$-extremal ray is a $K_X + (1-\eta)f^*H$-extremal ray for some small number $\eta > 0$. The length of the former extremal ray is less than or equal to that of the latter. Thus the result [8] of Kawamata for klt pairs gives the boudedness of the length of $K_X + f^*H$-extremal rays. By using the argument in [9], we have $\nu := \inf\{\frac{(K_X+f^*H+E,C)}{-(K_X+f^*H,C)} | \text{C is an extremal rational curve for } K_X + f^*H\} > 0$. Thus $K_X + f^*H + E + \nu(K_X + f^*H)$ is nef. □

**Assumption 3.3.** Furthermore assume that $K_Y$ is $\mathbf{Q}$-Cartier, that $-E$ is $f$-ample, and that, in the case where $f(E)$ is not a point, the divisor $(K_Y + H)|_{f(E)}$ is ample.

*Remark* 3.4. If $Y$ is $\mathbf{Q}$-factorial, then the condition that $-E$ is $f$-ample in Assumption 3.3 is automatically satisfied, under Assumption 3.1. (cf. Kollár-Mori [11], Lemma 2.62)

**Definition 3.5.** Under Assumptions 3.1 and 3.3. We define the number $\lambda$ by the equation $K_X + f^*H + E = f^*(K_Y + H) + (1+\lambda)E$. Then $1+\lambda < 0$, because $K_X + f^*H + E$ is nup.

**Proposition 3.6.** *Under Assumptions 3.1 and 3.3. The divisor $K_X + f^*H + E$ is big.*

*Proof.* Assume that $K_X + f^*H + (1-\epsilon)E = f^*(K_Y + H) + (1+\lambda-\epsilon)E$ is not big for every small number $\epsilon > 0$. Thus its self intersection number is zero for every $\epsilon$ from Proposition 3.2. Therefore $(-E)^{\dim E - \dim f(E)} \cdot (f^*(K_Y + H)^{\dim f(E)} \cdot E) = 0$. This contradicts to the $f$-ampleness of $-E$. Consequently $K_X + f^*H + (1-\epsilon)E$ is big for every small number $\epsilon > 0$ and so is $K_X + f^*H + (1-\epsilon)E + \epsilon E$. □

**Proposition 3.7.** *Under Assumptions 3.1 and 3.3. The divisor $(K_X + f^*H + E)|_E$ is ample.*

*Proof.* The divisor $f^*(K_Y + H)|_E - \epsilon E|_E$ on $E$ is ample for every small number $\epsilon > 0$ (cf. [11], Proposition 1.45). We also recall that $f^*(K_Y + H) + (1+\lambda-\epsilon)E = K_X + f^*H + (1-\epsilon)E$ is nef by Proposition 3.2. Thus $(K_X + f^*H + E)|_E = (f^*(K_Y + H) + (1+\lambda)E)|_E$ is ample, from the inequality $-\epsilon > 1+\lambda > 1+\lambda-\epsilon$. □

We state the main result of this section:

**Theorem 3.8.** *Under Assumptions 3.1 and 3.3. The divisor $K_X + f^*H + E$ is ample if and only if one of the following is satisfied:*
*(1)* $H = 0$
*(2)* $H \neq 0$ *and* $f^*H \cap E \neq \emptyset$
*(3)* $H \neq 0$, $f^*H \cap E = \emptyset$ *and* $((K_Y + H)|_H)^{n-1} > 0$.

For proof, we need the concept of "nef and log big".

**Definition 3.9.** Let $X$ be an $n$-dimensional nonsingular projective variety and $\Delta = \sum_{i\in I} \Delta_i$ a reduced simply normal crossing divisor on $X$ (where $\Delta_i$ is a prime divisor).

We define the strata $\mathbf{Strata}(\Delta) := \{\Gamma \mid \Gamma$ is an irreducible component of $\bigcap_{j\in J} \Delta_j \neq \emptyset$, for some nonempty subset $J$ of $I\}$ and the minimal strata $\mathbf{MS}(\Delta) := \{\Gamma \in \mathbf{Strata}(\Delta) \mid$ If $\Gamma' \in \mathbf{Strata}(\Delta)$ and $\Gamma' \subseteq \Gamma,$ then $\Gamma' = \Gamma\}$. We remark that $(K_X + \Delta) \mid_\Gamma = K_\Gamma$ for every $\Gamma \in \mathbf{MS}(\Delta)$.

Let $L$ be a Cartier divisor on $X$.

The divisor $L$ is said to be *nef and log big* on $(X, \Delta)$, if $L$ is nef, $L^n > 0$ and $(L \mid_\Gamma)^{\dim \Gamma} > 0$ for every $\Gamma \in \mathbf{Strata}(\Delta)$.

*Proof of Theorem 3.8.* The "only if " part is trivial. So we prove the "if" part.

In Case (1), $\mathbf{MS}(f^*H + E) = \{E\}$.

In Case (2), $\mathbf{MS}(f^*H + E) = \mathbf{MS}(f^*H|_E)$.

In Case (3), $\mathbf{MS}(f^*H + E) = \{f^*H, E\}$ and $(K_X + f^*H + E)|_{f^*H} \cong (K_Y + H)|_H$.

In any cases above, $((K_X+f^*H+E)|_\Gamma)^{\dim \Gamma} > 0$ for every $\Gamma \in \mathbf{MS}(f^*H+E)$ by Proposition 3.7. We also recall that $K_X+f^*H+E$ is big by Proposition 3.6

Consequently the base point free theorem [6] of Reid type implies that $K_X+f^*H+E$ is semiample, because it becomes nef and log big on $(X, f^*H+E)$.

□

*Example* 3.10. Let $\mathbf{P}^n$ $(n \geq 3)$ be a projective space with homogeneous coordinate $(x_0 : x_1 : x_2 : \cdots : x_n)$ and hyperplane $G$. We consider the hypersurface $Y$ $(\subset \mathbf{P}^n)$ defined by the irreducible homogeneous equation $x_1^l + x_2^l + x_3^l + \cdots + x_n^l = 0$ $(l \geq n+1)$. We note that $Y$ is normal and that $K_Y = (-(n+1)G + lG)|_Y = (l-(n+1))G|_Y$ is $\mathbf{Q}$-Cartier. Blow up $\mathbf{P}^n$ at the point $(1 : 0 : 0 : \cdots : 0)$ and obtain the morphism $\phi : \mathbf{P}' \to \mathbf{P}^n$ and the exceptional divisor $F$. Let $X$ be the strict transform of $Y$ by $\phi$. We note that $X$ is nonsingular. We have $K_{\mathbf{P}'} = \phi^*(-(n+1)G) + (n-1)F$. Thus $K_X = (K_{\mathbf{P}'} + X)|_X = (\phi^*(-(n+1)G) + (n-1)F + \phi^*(lG) - lF)|_X = (\phi^*(l-(n+1))G - (l-(n-1))F)|_X$. Then $K_X + (\phi^*G)|_X + F|_X = (l-n)(\phi^*G - F)|_X$ is nef from the theory of Seshadri constant. Consequently $K_X + (\phi^*(2G))|_X + F|_X = ((l-n)\phi^*G - (l-n)F + \phi^*G)|_X$ is nup because $-F$ is $\phi$-ample.

Let $H$ be the restriction of a general member of $|2G|$ to $Y$. We put $f := \phi|_X$. Then $E := \mathrm{Exc}(f) = F|_X$ is a smooth prime divisor and $-E = -F|_X$ is $f$-ample. We note that $K_X + f^*H + E$ is nup and that $(K_Y + H)|_H = (l - (n+1)+2)G|_H$ is ample. Lastly Theorem 3.8 (3) implies that $K_X + f^*H + E$ is ample.

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